

Motion with derivatives and integrals

AP Physics C (Mechanics and E&M, calculus-based) · Lesson

Quick answer: In calculus-based kinematics, velocity is the derivative of position and acceleration is the derivative of velocity. Going the other way, velocity is the integral of acceleration and position the integral of velocity, with constants set by initial conditions. These methods handle variable acceleration, where the constant-acceleration equations fail.

Level: AP Physics C (Mechanics and E&M, calculus-based) · **Curriculum:** AP PC:M Unit 1

How are position, velocity and acceleration linked by calculus?

Velocity is the rate of change of position, and acceleration is the rate of change of velocity. In calculus form, $v = dx/dt$ and $a = dv/dt = d^2x/dt^2$. Graphically, the slope of an x - t graph at an instant is the velocity, and the slope of a v - t graph is the acceleration. This works even when the acceleration is not constant, which is where the usual constant-acceleration equations fail.

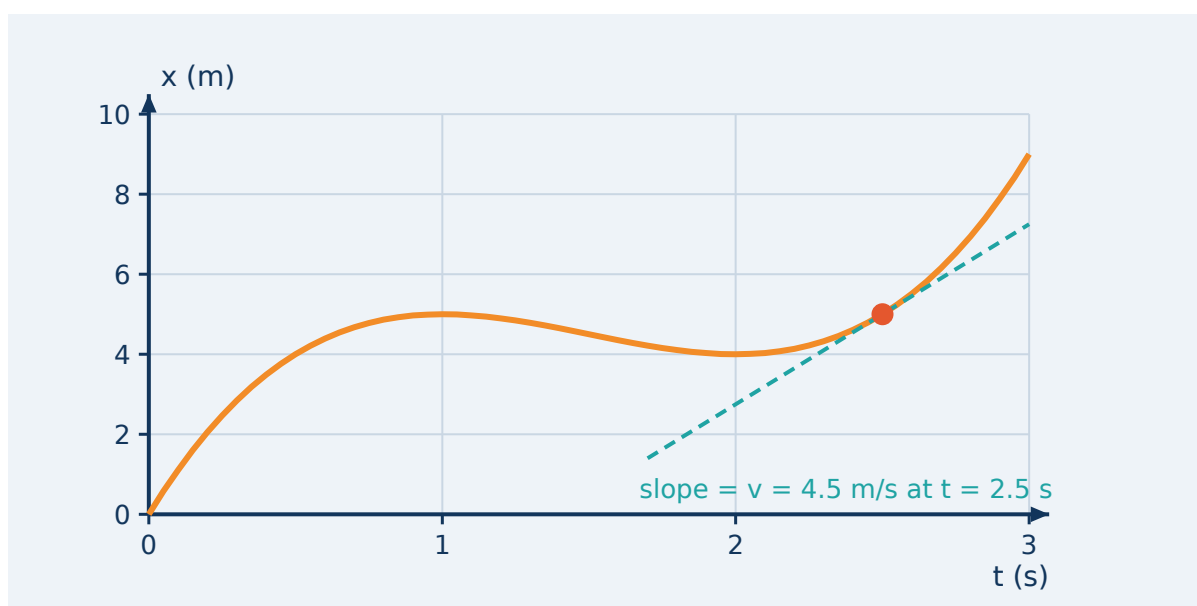
Definition

Instantaneous velocity is the limit of $\Delta x/\Delta t$ as Δt approaches zero, which is the derivative dx/dt .

Formula

$$v = dx/dt \quad a = dv/dt = d^2x/dt^2 \quad a = v \, dv/dx$$

x is position (m), v is velocity (m/s), a is acceleration (m/s^2) and t is time (s). The last form lets you work with acceleration that depends on position.



The slope of the x - t curve at any instant is the velocity.

Worked example

A particle moves with $x = 2t^3 - 9t^2 + 12t$ (m, with t in s). Method: $v = dx/dt = 6t^2 - 18t + 12$ and $a = 12t - 18$. The particle is at rest when $6(t - 1)(t - 2) = 0$, so at $t = 1$ s and $t = 2$ s. Positions: $x(1) = 5$ m and $x(2) = 4$ m. Over 0 to 3 s, x goes $0 \rightarrow 5 \rightarrow 4 \rightarrow 9$ m. Answer: displacement **9 m** but distance travelled **11 m**.

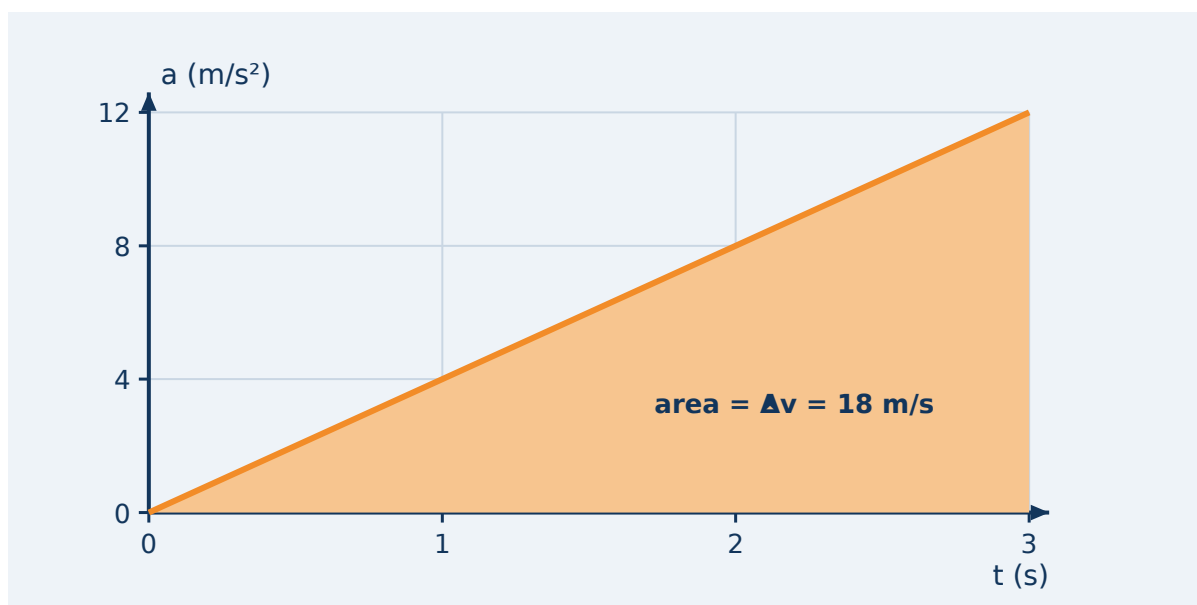
t (s)	x (m)	v (m/s)	a (m/s ²)
0	0	12	-18
1	5	0	-6
2	4	0	6
3	9	12	18

How do you go from acceleration back to velocity and position?

Integration reverses differentiation. The change in velocity is the area under the a - t graph, and the change in position is the area under the v - t graph. The constants of integration are fixed by the initial conditions, v_0 and x_0 .

Key law

$v = v_0 + \int a \, dt$ $x = x_0 + \int v \, dt$. The area under a v - t graph gives *displacement*, and area below the axis counts as negative.



The area under an a - t graph is the change in velocity.

Worked example

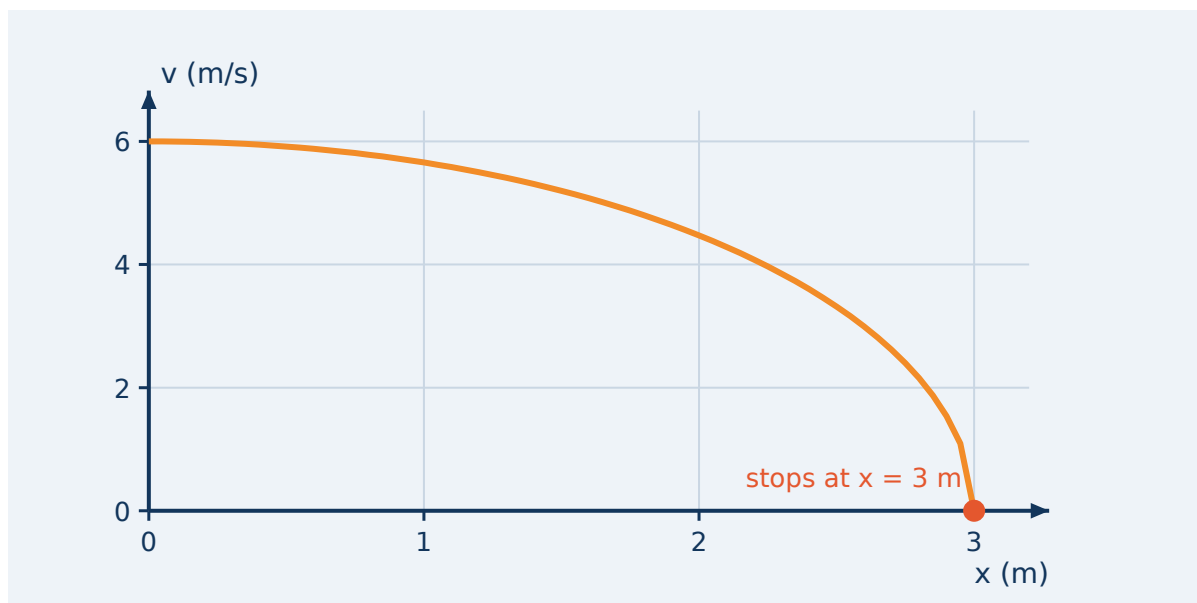
A cart has $a = 4t$ (m/s^2), $v_0 = 3$ m/s and $x_0 = 2$ m. Method: $v = 3 + \int 4t \, dt = 3 + 2t^2$, then $x = 2 + \int (3 + 2t^2) \, dt = 2 + 3t + (2/3)t^3$. At $t = 3$ s: $v = 3 + 18 = \mathbf{21 \text{ m/s}}$ and $x = 2 + 9 + 18 = \mathbf{29 \text{ m}}$.

Common mistake

Using $v = v_0 + at$ or $x = x_0 + v_0t + \frac{1}{2}at^2$ when a changes with time. These equations come from integrating a *constant*. Another slip is to forget the constants v_0 and x_0 after integrating.

What if the acceleration depends on position?

Sometimes a is given as a function of x , such as a spring-like force. Use the chain rule: $a = dv/dt = (dv/dx)(dx/dt) = v dv/dx$. Separate the variables and integrate: $\int v dv = \int a dx$.



Speed against position for $a = -4x$: the particle stops at $x = 3$ m.

Worked example

A mass has $a = -4x$ (with x in m) and passes $x = 0$ at 6 m/s. Method: $\int v dv = \int -4x dx$, so $v^2/2 = -2x^2 + C$. With $v = 6$ at $x = 0$, $C = 18$, so $v^2 = 36 - 4x^2$. It stops when $4x^2 = 36$. Answer: **$x = 3$ m**, where the acceleration is -12 m/s².

Lina's tip

Check the units after each derivative or integral. Differentiating divides the unit by seconds, and integrating multiplies it by seconds. If a result for velocity ends in m/s², you lost a time factor.

Key points to remember

- $v = dx/dt$ and $a = dv/dt = d^2x/dt^2$; slopes of x - t and v - t graphs.
- $v = v_0 + \int a dt$ and $x = x_0 + \int v dt$; areas under a - t and v - t graphs.
- If a depends on x , use $a = v dv/dx$ and integrate $\int v dv = \int a dx$.
- Displacement is the integral of v ; distance is the integral of $|v|$.
- Constant-acceleration equations only work when a is constant.

Motion with derivatives and integrals: frequently asked questions

What is the difference between displacement and distance in calculus?

Displacement is the integral of velocity and can be negative. Distance is the integral of speed $|v|$ and is never negative. To find distance, split the motion where velocity changes sign and add the pieces.

How do I find when a particle is at rest?

Set $v = dx/dt$ equal to zero and solve for t . Then check the position and the acceleration at those times. A particle can be momentarily at rest while still accelerating, such as at the top of a throw.

When does speed increase?

Speed increases when velocity and acceleration have the same sign, and decreases when they have opposite signs. Check both signs at the instant, not just whether acceleration is positive.

Why is the constant of integration important?

Each integral gives a family of functions that differ by a constant. The initial conditions v_0 and x_0 select the one that matches the real motion, so skipping them gives wrong values at every time.

What does $a = v dv/dx$ mean?

It is the chain rule applied to acceleration. It lets you solve for velocity as a function of position when a is given as a function of position, without first finding $x(t)$.