

Centre of mass, impulse and momentum with calculus

AP Physics C (Mechanics and E&M, calculus-based) · Lesson

Quick answer: The centre of mass is the mass-weighted average position, $x_{\text{cm}} = (1/M) \int x \, dm$. Impulse is the integral of force over time, $J = \int F \, dt$, and equals the change in momentum. External forces alone change the total momentum, $F_{\text{ext}} = dP/dt$, so internal forces cannot move the centre of mass.

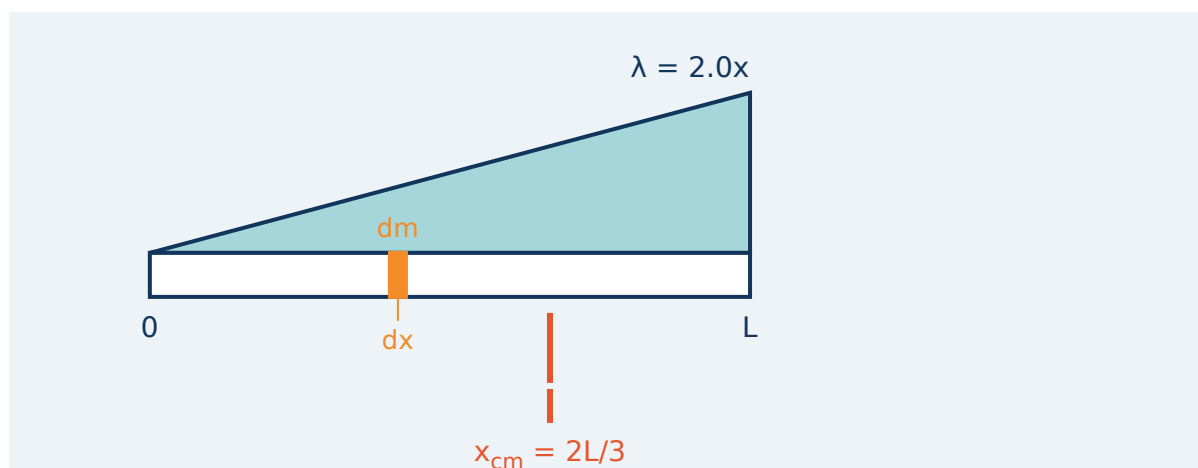
Level: AP Physics C (Mechanics and E&M, calculus-based) · **Curriculum:** AP PC:M Unit 4

How do you find the centre of mass by integration?

For separate masses the centre of mass is the mass-weighted mean position, $x_{\text{cm}} = \sum m_i x_i / \sum m_i$. For a continuous body, cut it into small pieces dm and let the sum become an integral.

Definition

The centre of mass of a system is the point whose position is the mass-weighted average of all positions: $r_{\text{cm}} = (1/M) \int r \, dm$. For a rod with linear density $\lambda(x)$, $dm = \lambda \, dx$ and $x_{\text{cm}} = (1/M) \int x \lambda \, dx$, where $M = \int \lambda \, dx$.



A rod with density increasing along its length has its centre of mass beyond the midpoint.

Worked example 1

Ananya in Pune models a tapering wooden pole of length $L = 1.2 \, \text{m}$ with density $\lambda = 2.0x \, \text{kg/m}$ (x measured from the thin end). $M = \int_0^L 2.0x \, dx = L^2 = 1.44 \, \text{kg}$. $\int x \lambda \, dx = \int_0^L 2.0x^2 \, dx = (2/3)L^3$. So $x_{\text{cm}} = (2/3)L^3/L^2 = (2/3)L = \mathbf{0.80 \, \text{m}}$ from the thin end.

Uniform body	Centre of mass position
Rod of length L	$L/2$ from either end

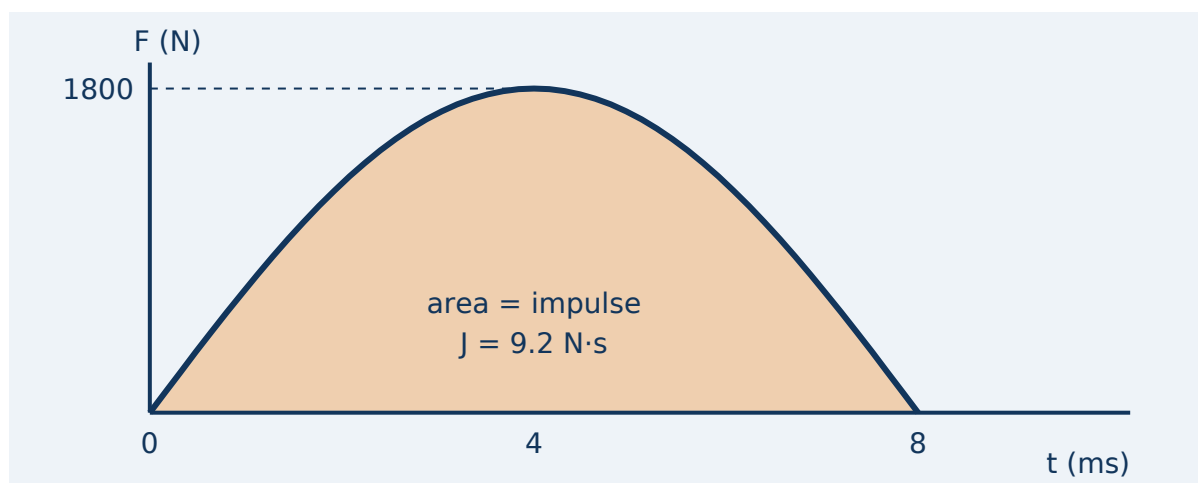
Uniform body	Centre of mass position
Triangular plate of height h	$h/3$ above the base
Semicircular plate of radius R	$4R/(3\pi)$ from the diameter
Solid hemisphere of radius R	$3R/8$ from the flat face

What is impulse and how does it change momentum?

Momentum is $p = mv$. Newton's second law in its general form says $F = dp/dt$. Integrating over a time interval gives the impulse-momentum theorem.

Formula

$J = \int F dt = \Delta p = mv_2 - mv_1$. J is impulse (N·s), F the force (N), t time (s), p momentum (kg·m/s). Impulse is the area under a force-time graph.



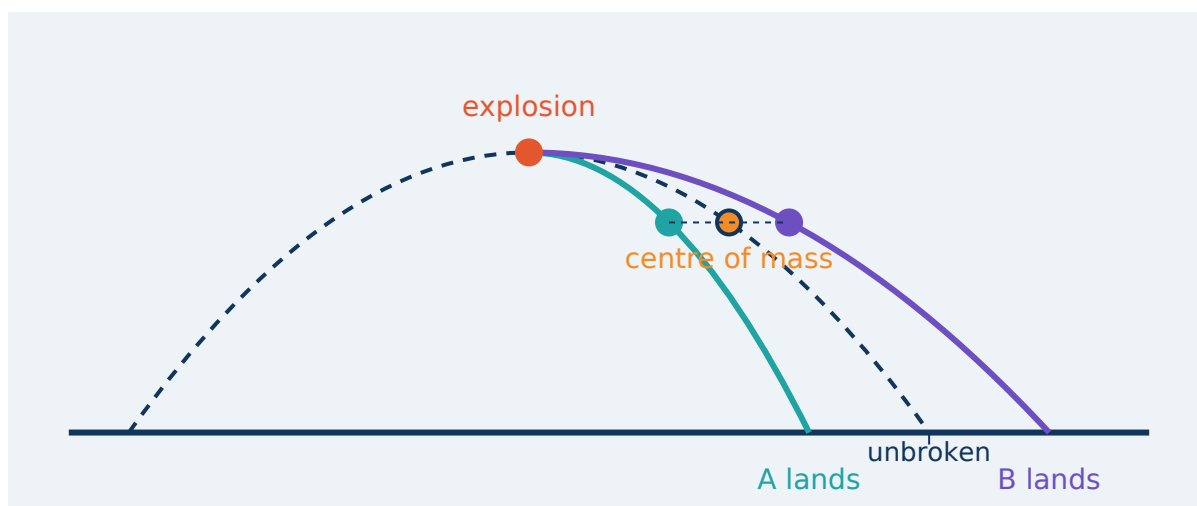
A kick modelled as a half-sine force pulse: the shaded area is the impulse delivered to the ball.

Worked example 2

Diego in Texas kicks a 0.43 kg football at rest. The force pulse is $F = F_0 \sin(\pi t/T)$ with $F_0 = 1800$ N and $T = 8.0$ ms. $J = \int_0^T F_0 \sin(\pi t/T) dt = 2F_0T/\pi = 2 \times 1800 \times 0.0080/\pi = 9.2$ N·s. The launch speed is $v = J/m = 9.2/0.43 = \mathbf{21 \text{ m/s}}$.

How does the centre of mass move?

Summing Newton's second law over all particles, the internal forces cancel in pairs (third law). What remains is $F_{\text{ext}} = M a_{\text{cm}} = dP/dt$. The centre of mass moves as if all the mass were there and only external forces acted. If $F_{\text{ext}} = 0$, the total momentum P is constant.



An explosion changes the motion of the pieces, not the path of their centre of mass.

When the mass of the system changes, apply $F_{\text{ext}} = dP/dt$ to the whole system, including the part that leaves or joins. For a rocket with exhaust speed v_e relative to the rocket and no external force, $m dv = -v_e dm$, which integrates to $\Delta v = v_e \ln(m_0/m)$.

Common mistake

Impulse is the area under a force-time graph, not a force-position graph. Momentum is a vector, so collisions need components, and kinetic energy is conserved only in elastic collisions. Momentum is conserved in all collisions where external impulse is negligible.

Lina's tip

Choose the origin at one end or at a massive piece to simplify x_{cm} . Use symmetry first: if a body has a plane of symmetry, the centre of mass lies in it.

Why do airbags and padded gloves work?

The impulse needed to stop an object is fixed by its change of momentum. Spreading it over a longer time makes the average force smaller, $F = J/\Delta t$. That is why a cricket fielder pulls the hands back when catching.

Key points to remember

- $x_{\text{cm}} = (1/M) \int x dm = (1/M) \int x \lambda dx$
- Impulse $J = \int F dt = \Delta p$ (area under F-t graph)
- $F_{\text{ext}} = M a_{\text{cm}} = dP/dt$
- Momentum is conserved when net external force is zero
- Rocket: $\Delta v = v_e \ln(m_0/m)$

Centre of mass, impulse and momentum with calculus:

frequently asked questions

How do you find the centre of mass of a non-uniform rod?

Write $dm = \lambda dx$, find the total mass $M = \int \lambda dx$, then compute $x_{cm} = (1/M) \int x \lambda dx$ over the length of the rod.

What is impulse?

Impulse is the integral of force over time, $J = \int F dt$, and equals the change in momentum Δp . It is the area under a force-time graph and has units $N \cdot s$ or $kg \cdot m/s$.

When is momentum conserved?

The total momentum of a system is constant when the net external force on it is zero, or when external impulses are negligible during a short collision or explosion.

Is kinetic energy conserved in collisions?

Only in elastic collisions. In inelastic collisions some kinetic energy becomes thermal energy, sound or deformation, while momentum is still conserved.

How does the rocket equation work?

With no external force, ejected exhaust carries momentum backward and the rocket gains forward momentum. This gives $\Delta v = v_e \ln(m_0/m)$, where m_0 is the initial mass and m the final mass.