

Moment of inertia by integration and torque

AP Physics C (Mechanics and E&M, calculus-based) · Lesson

Quick answer: Moment of inertia is $I = \int r^2 dm$, the sum of each mass element times the square of its distance from the rotation axis. The parallel axis theorem gives $I = I_{cm} + Md^2$ for a parallel axis, and torque produces angular acceleration through $\Sigma \tau = I\alpha$.

Level: AP Physics C (Mechanics and E&M, calculus-based) · **Curriculum:** AP PC:M Unit 5

What is moment of inertia?

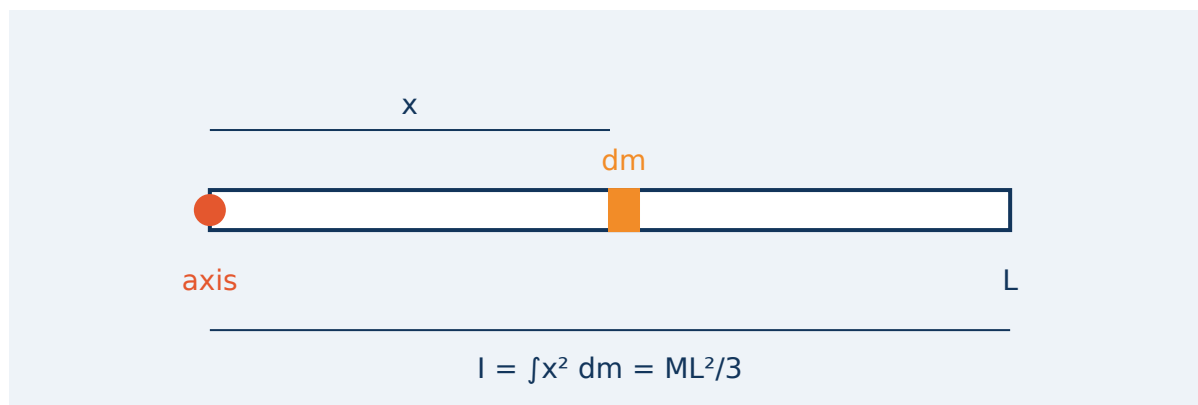
Moment of inertia measures how strongly a body resists changes in its rotation about a chosen axis. Each piece of mass contributes its mass times the square of its distance from the axis, so mass far from the axis counts much more.

Definition

The moment of inertia of a body about an axis is $I = \Sigma m_i r_i^2$, which becomes $I = \int r^2 dm$ for a continuous body. Here r is the perpendicular distance from the axis to the mass element dm . The unit is $\text{kg}\cdot\text{m}^2$.

How do you find I by integration?

Choose an element dm whose points are all at the same distance from the axis, write dm in terms of one variable, then integrate. For a thin rod of mass M and length L , $dm = (M/L) dx$.



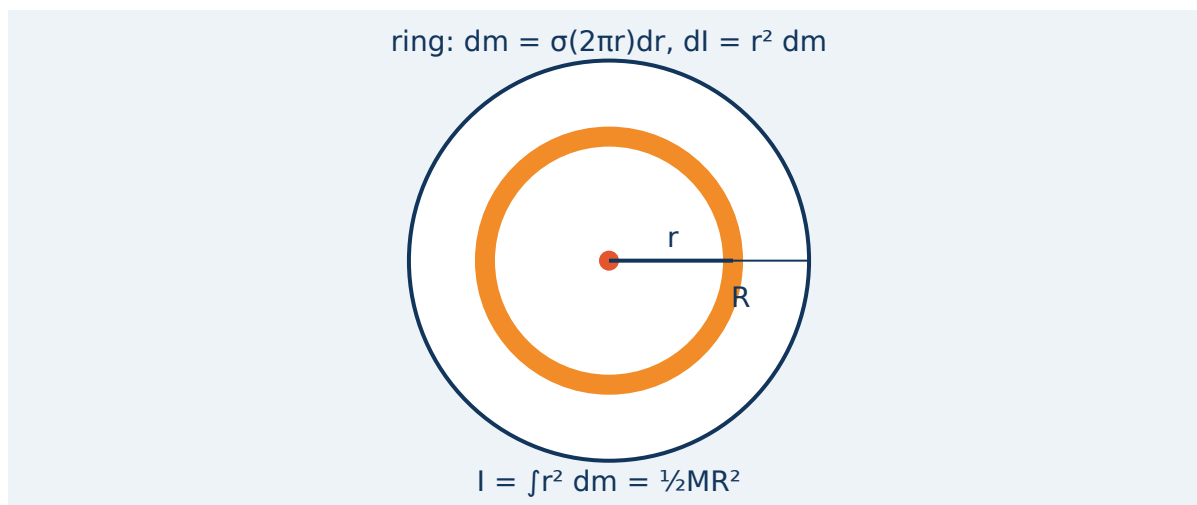
A thin rod rotating about one end: each element dm sits a different distance x from the axis.

About an axis through one end: $I = \int_0^L x^2 (M/L) dx = ML^2/3$. About the centre, integrate from $-L/2$ to $L/2$ to get $ML^2/12$.

Worked example 1

Ananya in Pune models a rod for a school windmill: $M = 0.60 \text{ kg}$ and $L = 1.0 \text{ m}$. About one end, $I = ML^2/3 = 0.60 \times 1.0^2/3 = \mathbf{0.20 \text{ kg}\cdot\text{m}^2}$. About its centre, $I = ML^2/12 = 0.60 \times 1.0^2/12 = 0.050 \text{ kg}\cdot\text{m}^2$, four times smaller, because the mass sits closer to the axis.

For a uniform disc, use thin rings of radius r and thickness dr . The ring has area $2\pi r dr$ and mass $dm = \sigma(2\pi r dr)$ where $\sigma = M/\pi R^2$. Then $I = \int_0^R r^2 \sigma 2\pi r dr = \sigma \pi R^4/2 = \frac{1}{2}MR^2$.



A solid disc is built from rings, each contributing $r^2 dm$.

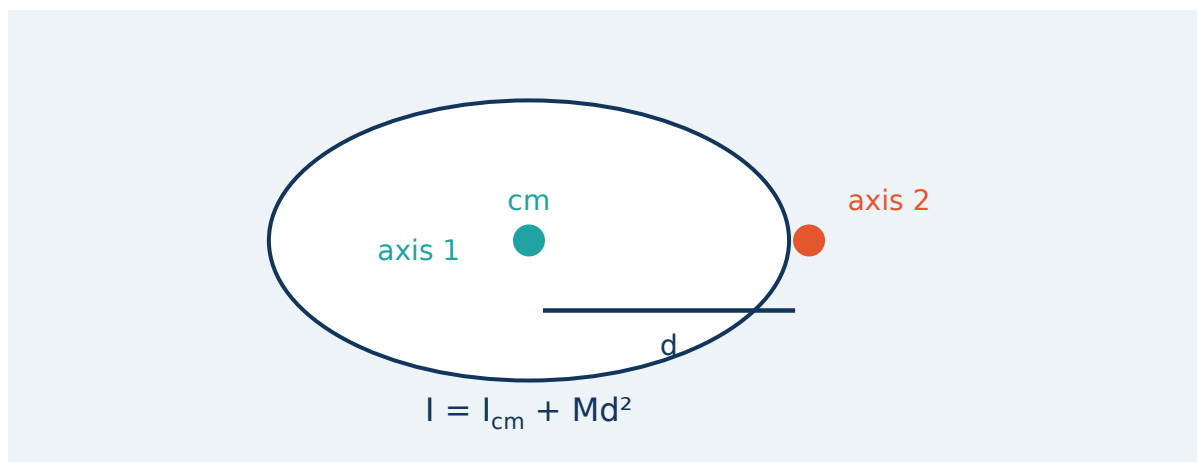
Body (uniform)	Axis	I
Thin hoop	Central, perpendicular to plane	MR^2
Solid disc or cylinder	Central axis	$\frac{1}{2}MR^2$
Solid sphere	Through the centre	$(2/5)MR^2$
Thin spherical shell	Through the centre	$(2/3)MR^2$
Thin rod	Through the centre / one end	$ML^2/12$ / $ML^2/3$

What does the parallel axis theorem say?

If you know I_{cm} about an axis through the centre of mass, the moment of inertia about any parallel axis at distance d is larger by Md^2 .

Formula

$I = I_{cm} + Md^2$. I is the moment of inertia about the new axis ($\text{kg}\cdot\text{m}^2$), I_{cm} that about the parallel axis through the centre of mass, M the total mass (kg) and d the distance between the axes (m).



The parallel axis theorem links axes with the same direction.

Check with the rod: $I = ML^2/12 + M(L/2)^2 = ML^2/3$, which matches the integral. For a disc of $M = 2.0$ kg and $R = 0.20$ m, the axis through a point on the rim gives $I = \frac{1}{2}MR^2 + MR^2 = 1.5 \times 2.0 \times 0.040 = 0.12$ kg·m².

How do torque and angular acceleration connect?

The torque of a force is $\tau = rF \sin \theta$, where θ is the angle between the force and the position vector from the axis (so $rF \sin \theta$ is force times lever arm). Newton's second law for rotation reads $\Sigma \tau = I\alpha$, and the angular momentum is $L = I\omega$.

Worked example 2

Kofi in Accra winds a rope around a flywheel (solid disc, $M = 4.0$ kg, $R = 0.25$ m) and pulls it with a steady 10 N. $\tau = RF = 0.25 \times 10 = 2.5$ N·m. $I = \frac{1}{2}MR^2 = \frac{1}{2} \times 4.0 \times 0.0625 = 0.125$ kg·m². $\alpha = \tau/I = \mathbf{20 \text{ rad/s}^2}$. After 3.0 s from rest, $\omega = 60$ rad/s and $K = \frac{1}{2}I\omega^2 = 225$ J.

Common mistake

Use the perpendicular distance from the axis, not the distance from an origin. The parallel axis theorem must start from an axis through the centre of mass: you cannot add Md^2 to the moment of inertia about an end.

Lina's tip

Learn the rod, disc and sphere results, then build others by adding pieces. A compound body has I equal to the sum of the I values of its parts, each about the same axis.

Key points to remember

- $I = \int r^2 dm$; unit kg·m²
- Rod: $ML^2/12$ (centre), $ML^2/3$ (end); disc $\frac{1}{2}MR^2$; sphere $(2/5)MR^2$
- Parallel axis theorem: $I = I_{cm} + Md^2$
- Torque $\tau = rF \sin \theta$; $\Sigma \tau = I\alpha$
- Angular momentum $L = I\omega$; $K_{rot} = \frac{1}{2}I\omega^2$

Moment of inertia by integration and torque: frequently asked questions

What is the formula for moment of inertia?

$I = \sum m_i r_i^2$ for separate masses, and $I = \int r^2 dm$ for a continuous body. The distance r is measured perpendicular to the rotation axis.

How do you find moment of inertia by integration?

Choose a mass element dm at one distance from the axis, write dm in terms of a single variable such as dx or dr , then integrate $r^2 dm$ over the whole body.

What does the parallel axis theorem state?

$I = I_{cm} + Md^2$, where I_{cm} is the moment of inertia about a parallel axis through the centre of mass and d is the distance between the two axes.

What is the rotational form of Newton's second law?

The net torque about a fixed axis equals the moment of inertia times angular acceleration, $\Sigma \tau = I\alpha$. It plays the same role as $F = ma$.

Why does a hoop roll more slowly than a disc?

The hoop has all its mass at the rim, so I is larger (MR^2 against $\frac{1}{2}MR^2$). More of the energy goes into rotation and less into translation, giving a smaller speed.