

# Moment of inertia by integration and torque

AP Physics C (Mechanics and E&M, calculus-based) · Worksheet answers

Each answer gives the principle, the substitution and the final result with units. The derivations show every integration step, so check your own working against them line by line. Use  $g = 9.81 \text{ m/s}^2$  unless told otherwise, and keep two or three significant figures.

**Level:** AP Physics C (Mechanics and E&M, calculus-based) · **Curriculum:** AP PC:M Unit 5

## Problem 1 Dumbbell inertia

$$I = \Sigma mr^2 = 2 \times 0.50 \times (0.40)^2 = \mathbf{0.16 \text{ kg}\cdot\text{m}^2}.$$

## Problem 2 Units

1.  $\text{kg}\cdot\text{m}^2$ .
2.  $\mathbf{N}\cdot\mathbf{m}$ . A joule is energy, a scalar; torque is a vector quantity with a different meaning, even though the dimensions match.

## Problem 3 Torque from a spanner

1.  $\tau = rF = 0.25 \times 30 = \mathbf{7.5 \text{ N}\cdot\text{m}}$ .
2.  $\tau = rF \sin 30^\circ = 7.5 \times 0.50 = \mathbf{3.8 \text{ N}\cdot\text{m}}$ .

## Problem 4 Angular acceleration

$$\alpha = \tau/I = 6.0/0.40 = \mathbf{15 \text{ rad/s}^2}.$$

## Problem 5 Using the table

1.  $I = (2/5)MR^2 = 0.40 \times 5.0 \times 0.010 = \mathbf{0.020 \text{ kg}\cdot\text{m}^2}$ .
2.  $I = MR^2 = 5.0 \times 0.010 = \mathbf{0.050 \text{ kg}\cdot\text{m}^2}$ .

## Problem 6 Ring against disc

1. The **ring** ( $MR^2$  against  $\frac{1}{2}MR^2$ ), since all its mass is at the largest radius.
2. The **disc** has the larger  $\alpha$ , because  $\alpha = \tau/I$  and its  $I$  is smaller.

## Problem 7 Rod about two axes

1.  $I_{\text{cm}} = ML^2/12 = 0.60/12 = \mathbf{0.050 \text{ kg}\cdot\text{m}^2}$ .
2.  $I = I_{\text{cm}} + M(L/2)^2 = 0.050 + 0.60 \times 0.25 = \mathbf{0.20 \text{ kg}\cdot\text{m}^2}$ , equal to  $ML^2/3$ .

### Problem 8 Disc about the rim

- $\frac{1}{2}MR^2 = \frac{1}{2} \times 2.0 \times 0.040 = \mathbf{0.040 \text{ kg}\cdot\text{m}^2}$ .
- $I = 0.040 + 2.0 \times (0.20)^2 = \mathbf{0.12 \text{ kg}\cdot\text{m}^2}$ .

### Problem 9 Rod with varying density

- $M = \int_0^{1.5} 2.0x^2 dx = (2.0/3)(1.5)^3 = \mathbf{2.25 \text{ kg}}$ .
- $I = \int x^2 dm = \int_0^{1.5} 2.0x^4 dx = (2.0/5)(1.5)^5 = 0.40 \times 7.594 = \mathbf{3.0 \text{ kg}\cdot\text{m}^2}$ .

### Problem 10 Flywheel on a rope

- $\tau = RF = 0.30 \times 12 = 3.6 \text{ N}\cdot\text{m}$ .
- $I = \frac{1}{2}MR^2 = \frac{1}{2} \times 6.0 \times 0.090 = 0.27 \text{ kg}\cdot\text{m}^2$ .
- $\alpha = 3.6/0.27 = \mathbf{13 \text{ rad/s}^2}$ .  $\omega = \alpha t = 13.3 \times 2.5 = \mathbf{33 \text{ rad/s}}$ .

### Problem 11 Skater spinning

- $L = I\omega$  is conserved:  $\omega_2 = 3.0 \times 4.0/1.2 = \mathbf{10 \text{ rad/s}}$ .
- $K_1 = \frac{1}{2} \times 3.0 \times 16 = 24 \text{ J}$ ;  $K_2 = \frac{1}{2} \times 1.2 \times 100 = 60 \text{ J}$ . The skater does **36 J** of work pulling her arms in.

### Problem 12 Reading $\omega$ against $t$

- $\alpha = 10/4.0 = \mathbf{2.5 \text{ rad/s}^2}$ .
- $\tau = I\alpha = 0.50 \times 2.5 = \mathbf{1.25 \text{ N}\cdot\text{m}}$ .
- $\theta = \text{area} = \frac{1}{2} \times 4.0 \times 10 = \mathbf{20 \text{ rad}}$ .

### Problem 13 Plank pivot point

- The centre is at 1.2 m, so  $d = 1.2 - 0.40 = \mathbf{0.80 \text{ m}}$ .
- $I = ML^2/12 + Md^2 = 3.84 + 8.0 \times 0.64 = 3.84 + 5.12 = \mathbf{9.0 \text{ kg}\cdot\text{m}^2}$ .
- The theorem applies only when one axis passes through the centre of mass;  $ML^2/3$  is the value about an end, so that sum is invalid (it gives  $16.6 \text{ kg}\cdot\text{m}^2$ ).

### Problem 14 Swinging signboard bar

- $I = ML^2/3 = 2.5 \times 2.56/3 = \mathbf{2.1 \text{ kg}\cdot\text{m}^2}$ .
- $\tau = Mg(L/2) = 2.5 \times 9.81 \times 0.80 = 19.6 \text{ N}\cdot\text{m}$ , so  $\alpha = 19.6/2.13 = \mathbf{9.2 \text{ rad/s}^2}$ .
- The centre accelerates downward at  $a = \alpha(L/2) = 7.36 \text{ m/s}^2$ . Then  $Mg - F = Ma$  gives  $F = 2.5 \times (9.81 - 7.36) = \mathbf{6.1 \text{ N}}$  (one quarter of the weight).

### Problem 15 Disc by integration

- $\sigma = M/(\pi R^2)$ ,  $dm = \sigma 2\pi r dr = (2M/R^2) r dr$ .

$$2. I = \int_0^R r^2 (2M/R^2) r dr = (2M/R^2)(R^4/4) = \frac{1}{2}MR^2.$$

$$3. I = \frac{1}{2} \times 0.80 \times 0.0225 = \mathbf{9.0 \times 10^{-3} \text{ kg}\cdot\text{m}^2}.$$

### Problem 16 Annular disc

$$1. \sigma = M/[\pi(R_2^2 - R_1^2)]. I = \int_{R_1}^{R_2} r^2 \sigma 2\pi r dr = (\sigma\pi/2)(R_2^4 - R_1^4) = (\sigma\pi/2)(R_2^2 - R_1^2)(R_2^2 + R_1^2) = \frac{1}{2}M(R_1^2 + R_2^2).$$

$$2. I = \frac{1}{2} \times 3.0 \times (0.010 + 0.040) = \mathbf{0.075 \text{ kg}\cdot\text{m}^2}.$$

### Problem 17 Massive pulley

$$1. m_1g - T_1 = m_1a; T_2 - m_2g = m_2a; (T_1 - T_2)R = I\alpha = \frac{1}{2}MR^2 \times a/R, \text{ so } T_1 - T_2 = \frac{1}{2}Ma.$$

$$\text{Adding: } (m_1 - m_2)g = (m_1 + m_2 + M/2)a, \text{ so } a = 9.81/5.5 = \mathbf{1.8 \text{ m/s}^2}.$$

$$2. T_1 = 3.0(9.81 - 1.78) = \mathbf{24 \text{ N}}; T_2 = 2.0(9.81 + 1.78) = \mathbf{23 \text{ N}}.$$

### Problem 18 Rolling cylinder

$$1. \text{ Along the slope: } Mg \sin \theta - f = Ma; \text{ torque: } fR = I\alpha = Ia/R. \text{ Eliminating } f: a = g \sin \theta / (1 + I/MR^2) = 9.81 \times 0.50/1.5 = \mathbf{3.3 \text{ m/s}^2}.$$

$$2. v = \sqrt{2as} = \sqrt{2 \times 3.27 \times 2.0} = \mathbf{3.6 \text{ m/s}}.$$

### Problem 19 Compound pendulum arm

$$1. I = ML^2/3 + mL^2 = 0.20 + 0.30 = \mathbf{0.50 \text{ kg}\cdot\text{m}^2}.$$

$$2. \tau = (M \times 0.50 + m \times 1.0)g = (0.30 + 0.30) \times 9.81 = 5.89 \text{ N}\cdot\text{m}, \text{ so } \alpha = 5.89/0.50 = \mathbf{12 \text{ rad/s}^2}.$$

### Problem 20 Clay hits a disc

$$1. L = mvR = 0.050 \times 6.0 \times 0.30 = \mathbf{0.090 \text{ kg}\cdot\text{m}^2/\text{s}}.$$

$$2. I = \frac{1}{2} \times 1.0 \times 0.090 + 0.050 \times 0.090 = 0.0495 \text{ kg}\cdot\text{m}^2. \omega = L/I = \mathbf{1.8 \text{ rad/s}}.$$

$$3. K_1 = \frac{1}{2} \times 0.050 \times 36 = 0.90 \text{ J}; K_2 = \frac{1}{2} \times 0.0495 \times 3.31 = 0.082 \text{ J}; \text{ loss} = \mathbf{0.82 \text{ J}}.$$