

Kirchhoff's rules and RC transients

AP Physics C (Mechanics and E&M, calculus-based) · Lesson

Quick answer: Kirchhoff's junction rule conserves charge and his loop rule conserves energy, giving equations for every branch current. In an RC circuit the loop rule $\varepsilon - IR - q/C = 0$ leads to $q = C\varepsilon(1 - e^{(-t/RC)})$ when charging and $q = Q_0 e^{(-t/RC)}$ when discharging, with time constant $\tau = RC$.

Level: AP Physics C (Mechanics and E&M, calculus-based) · **Curriculum:** AP PC:E&M Unit 11

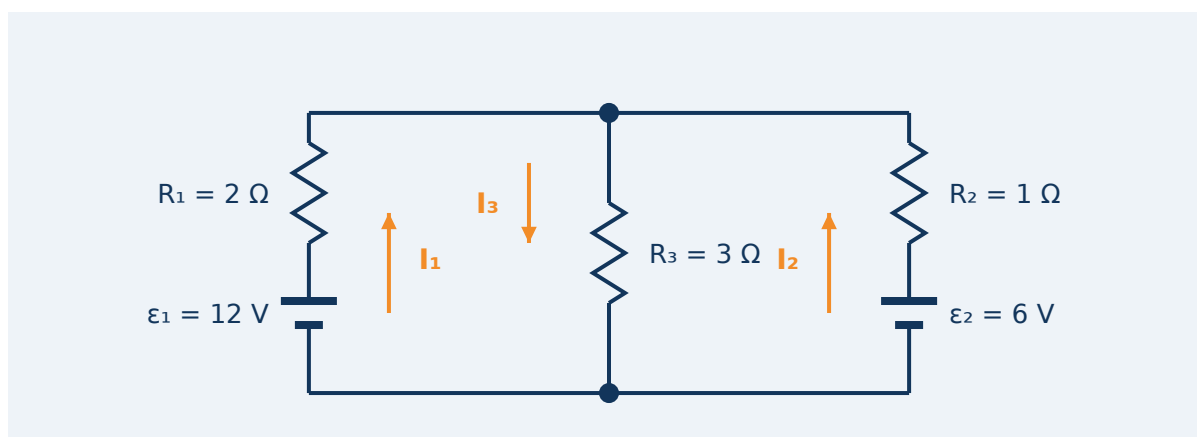
What are Kirchhoff's rules and when do you use them?

Ohm's law handles single resistors, but a circuit with several sources and branches needs two conservation laws. One is about charge, the other about energy, and together they give enough equations to find every current.

Key law

Junction rule: the sum of currents into a junction equals the sum out of it (conservation of charge). **Loop rule:** the sum of potential changes around any closed loop is zero (conservation of energy).

Assign a current direction in every branch. Going round a loop, a battery contributes $+\varepsilon$ when you cross from $-$ to $+$, a resistor contributes $-IR$ when you go with the current, and a capacitor contributes $-q/C$ when you cross it from the positive plate to the negative plate. If a solved current comes out negative, it simply flows opposite to your guess.



Two batteries sharing a middle resistor: one junction equation and two loop equations give I_1 , I_2 and I_3 .

Worked example

Data: $\varepsilon_1 = 12\text{ V}$, $R_1 = 2\ \Omega$, $\varepsilon_2 = 6\text{ V}$, $R_2 = 1\ \Omega$ and $R_3 = 3\ \Omega$ as in the figure. **Find** the three currents.

Method: $I_3 = I_1 + I_2$. Left loop: $12 - 2I_1 - 3I_3 = 0$, so $12 = 5I_1 + 3I_2$. Right loop: $6 - I_2 - 3I_3 = 0$, so $6 = 3I_1 + 4I_2$. Solve the pair.

Answer: $I_1 = 2.7 \text{ A}$, $I_2 = -0.55 \text{ A}$ (battery 2 is being charged) and $I_3 = 2.2 \text{ A}$.

How does a capacitor charge through a resistor?

Close the switch in a series circuit with a battery ϵ , resistor R and capacitor C . At every instant the loop rule gives $\epsilon - IR - q/C = 0$ with $I = dq/dt$. This is a differential equation: $dq/dt = (\epsilon - q/C)/R$. Separating variables and integrating from $q = 0$ gives an exponential approach to the final charge.

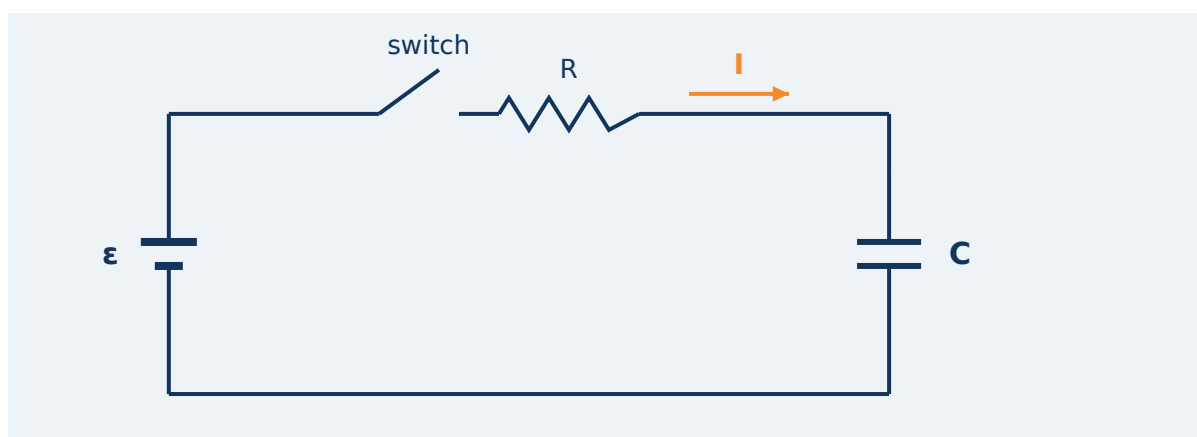
Formula

$$q(t) = C\epsilon(1 - e^{-t/\tau}) \quad I(t) = (\epsilon/R) e^{-t/\tau} \quad V_C = \epsilon(1 - e^{-t/\tau}) \quad \tau = RC$$

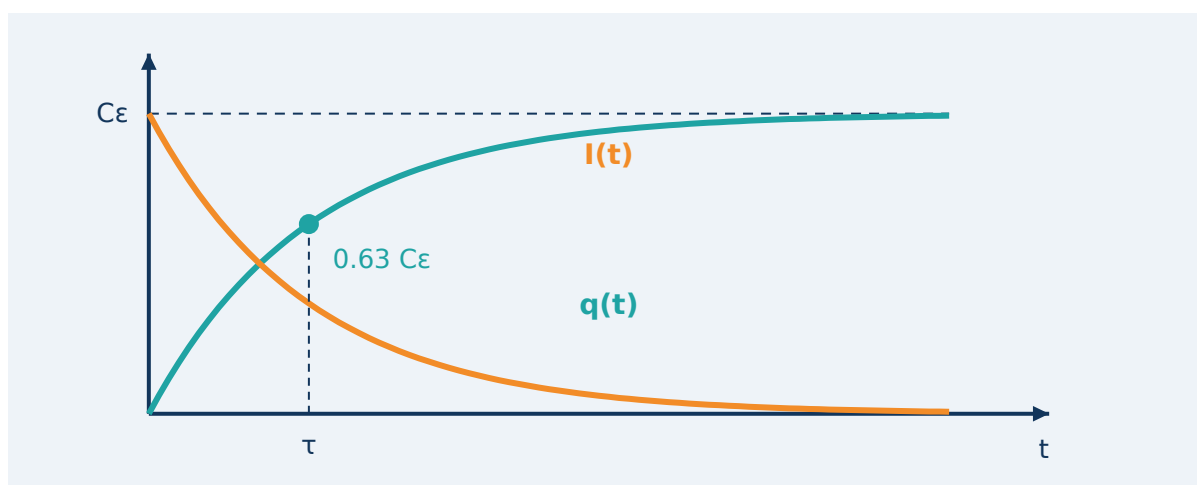
τ is the time constant (s) when R is in ohms and C is in farads. For discharge from Q_0 : $q = Q_0 e^{-t/\tau}$ and $I = (Q_0/\tau) e^{-t/\tau}$.

Definition

The **time constant** $\tau = RC$ is the time in which a charging capacitor reaches 63.2 % of its final charge, or a discharging one falls to 36.8 % of its initial charge.



A charging circuit: the loop rule gives $\epsilon - IR - q/C = 0$.



While charging, q rises towards $C\epsilon$ while I falls from ϵ/R ; both change by 63 % of the remaining gap in each time constant.

Worked example

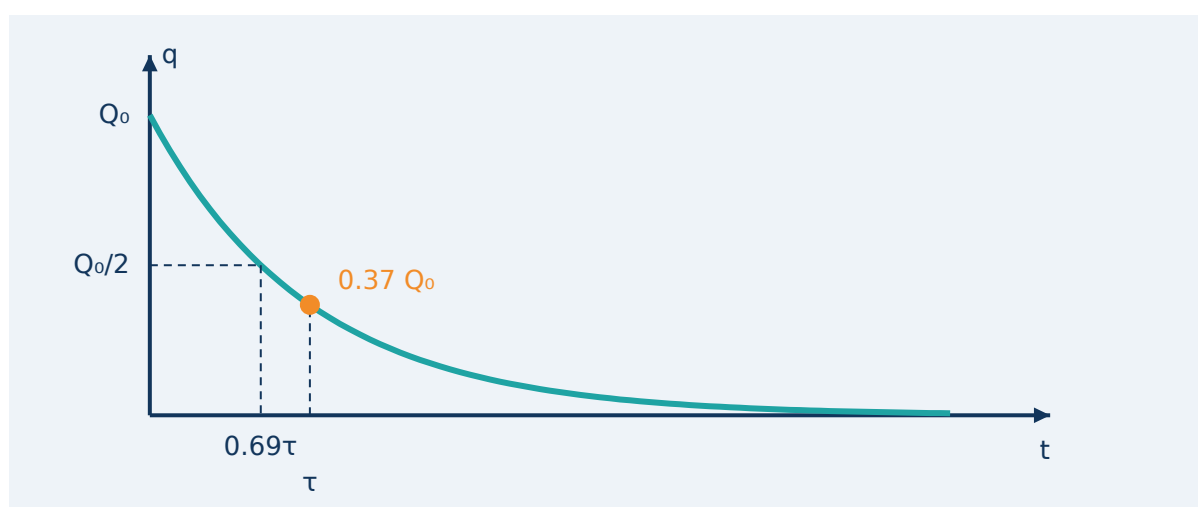
Data: $\varepsilon = 9.0 \text{ V}$, $R = 20 \text{ k}\Omega$ and $C = 47 \text{ }\mu\text{F}$, uncharged at $t = 0$. **Find** V_C and I at $t = 1.5 \text{ s}$.

Method: $\tau = RC = (2.0 \times 10^4)(47 \times 10^{-6}) = 0.94 \text{ s}$, so $e^{-t/\tau} = e^{-1.596} = 0.203$. Then $V_C = \varepsilon(1 - 0.203)$ and $I = (\varepsilon/R)(0.203)$.

Answer: $V_C = 7.2 \text{ V}$ and $I = 91 \text{ }\mu\text{A}$.

How does a capacitor discharge and what is the time constant?

If the battery is removed and the capacitor discharges through R , the loop rule becomes $q/C = IR$ with $I = -dq/dt$, so q decays as $Q_0 e^{-t/\tau}$. After one time constant 36.8 % of the charge remains, and after 0.693τ half of it remains, because $t_{1/2} = \tau \ln 2$.



During discharge q falls exponentially: 37 % remains after τ and 50 % after 0.69τ .

Time	Charging: $q/C\varepsilon$	Charging: $I/(\varepsilon/R)$	Discharging: q/Q_0
0	0	1	1
τ	0.632	0.368	0.368
2τ	0.865	0.135	0.135
3τ	0.950	0.050	0.050
5τ	0.993	0.007	0.007

Worked example

Data: a $100 \text{ }\mu\text{F}$ capacitor charged to 12 V discharges through $50 \text{ k}\Omega$. **Find** the time for V_C to fall to 3.0 V .

Method: $\tau = (5.0 \times 10^4)(1.0 \times 10^{-4}) = 5.0 \text{ s}$, and $3.0 = 12e^{-t/\tau}$ gives $t = \tau \ln 4$.

Answer: $t = 6.9 \text{ s}$.

Energy accounting is neat. A battery supplies $C\varepsilon^2$ while charging, half is stored in the capacitor ($\frac{1}{2}C\varepsilon^2$), and half is dissipated in the resistor, whatever the value of R . Integrating I^2R from 0 to infinity confirms the resistor share.

Common mistake

Do not say the capacitor is fully charged after one time constant. At $t = \tau$ it holds only 63 % of its final charge. Treat it as essentially full after about 5τ .

Lina's tip

An uncharged capacitor acts like a wire at $t = 0$, and a fully charged one acts like a break in the circuit. Solve those two limits first and the exponential fills the gap.

Key points to remember

- Junction rule: $\Sigma I_{\text{in}} = \Sigma I_{\text{out}}$. Loop rule: $\Sigma \Delta V = 0$ around any closed loop.
- Charging: $q = C\epsilon(1 - e^{-(t/\tau)})$, $I = (\epsilon/R)e^{-(t/\tau)}$, $\tau = RC$.
- Discharging: $q = Q_0 e^{-(t/\tau)}$; 36.8 % remains at $t = \tau$ and half at $t = 0.693\tau$.
- At $t = 0$ an uncharged capacitor is a wire; at $t \rightarrow \infty$ it is an open circuit.
- Charging from a battery: $C\epsilon$ supplied, $\frac{1}{2}C\epsilon^2$ stored, $\frac{1}{2}C\epsilon^2$ lost as heat in R.

Kirchhoff's rules and RC transients: frequently asked questions

What does the time constant RC mean?

It is the time for a charging capacitor to reach 63.2 % of its final charge, or for a discharging one to fall to 36.8 % of its start. A larger R or C makes the change slower.

Why does the current fall as a capacitor charges?

As q grows, the capacitor voltage q/C opposes the battery. The voltage across R, $\epsilon - q/C$, shrinks, so $I = (\epsilon - q/C)/R$ decays exponentially to zero.

Where does the energy go when a capacitor is charged through a resistor?

The battery supplies $C\epsilon^2$. Half is stored in the electric field as $\frac{1}{2}C\epsilon^2$ and the other half is dissipated as heat in the resistor, independent of the size of R.

How do I apply the loop rule with a capacitor?

Treat the capacitor as an element with potential difference q/C . Crossing it from its positive to its negative plate lowers the potential by q/C , so it enters like $-q/C$.

How long until a capacitor is considered fully charged?

After 5τ the charge is within 1 % of its final value, and after 7τ it is within 0.1 %. Engineers treat the circuit as steady after about five time constants.