

Simple harmonic motion and the physical pendulum

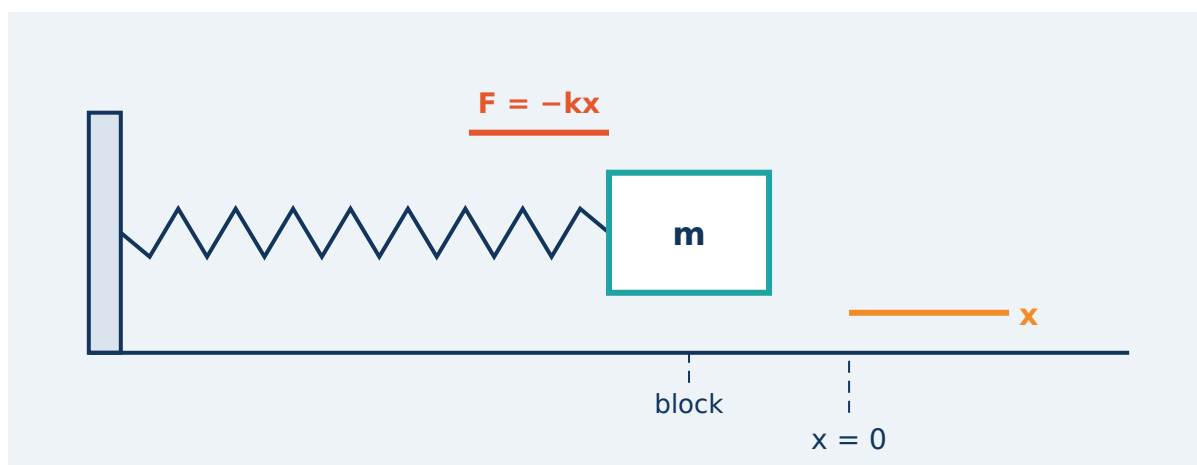
AP Physics C (Mechanics and E&M, calculus-based) · Lesson

Quick answer: Simple harmonic motion obeys $d^2x/dt^2 = -\omega^2x$, with solution $x = A \cos(\omega t + \phi)$. For a spring $\omega = \sqrt{k/m}$; for a physical pendulum at small angles $\omega = \sqrt{mgd/I}$. The period is $T = 2\pi/\omega$ and depends on the system, not on the amplitude.

Level: AP Physics C (Mechanics and E&M, calculus-based) · **Curriculum:** AP PC:M Unit 7

What makes motion simple harmonic?

Simple harmonic motion (SHM) happens when the net force or torque is proportional to the displacement and points back towards equilibrium. A mass on a spring is the model case: Hooke's law gives $F = -kx$, and Newton's second law then links the motion to calculus.



The restoring force $F = -kx$ always points towards $x = 0$.

Definition

An object undergoes simple harmonic motion when its acceleration is proportional to its displacement from equilibrium and opposite in direction: $d^2x/dt^2 = -\omega^2x$. The constant ω is the angular frequency in rad/s, with $T = 2\pi/\omega$ and $f = 1/T$.

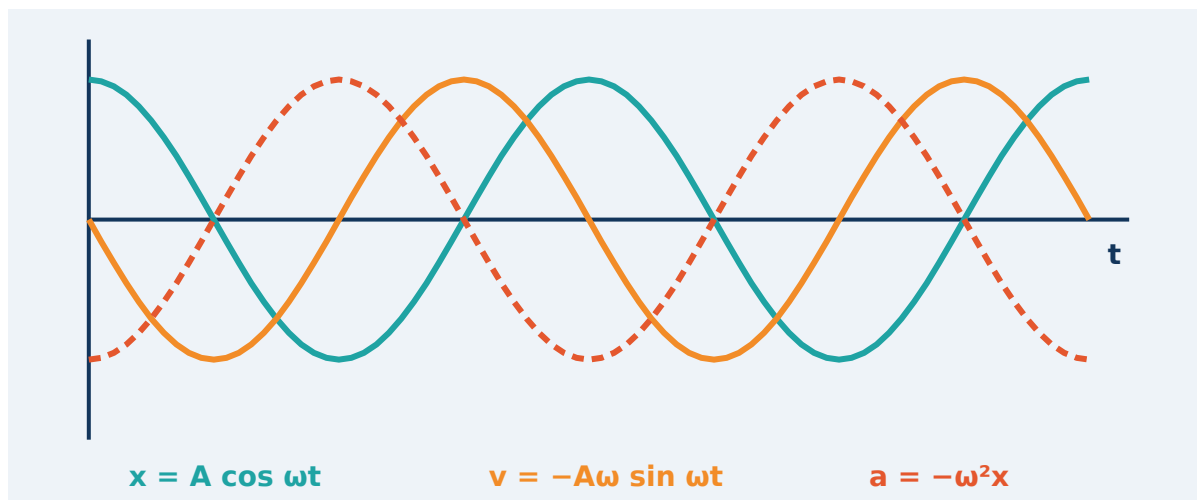
How does the differential equation give $x(t)$?

Newton's second law for the spring gives $m d^2x/dt^2 = -kx$, which rearranges to $d^2x/dt^2 + (k/m)x = 0$. A function whose second derivative is a negative multiple of itself is a sinusoid, so $x = A \cos(\omega t + \phi)$. Differentiating twice shows that it works if $\omega^2 = k/m$.

Formula

$x = A \cos(\omega t + \phi)$; $v = dx/dt = -A\omega \sin(\omega t + \phi)$; $a = d^2x/dt^2 = -\omega^2x$. A is the amplitude in

m , ϕ the phase constant in rad, $\omega = \sqrt{k/m}$ in rad/s. Maximum speed: $A\omega$. Maximum acceleration: $A\omega^2$. Total energy: $E = \frac{1}{2}kA^2$.



Displacement (teal), velocity (orange) and acceleration (red, dashed) over two periods.

The constants A and ϕ are not set by k and m . They come from the starting conditions $x(0)$ and $v(0)$. That is why the period of a spring does not change when you pull the block further.

Worked example 1

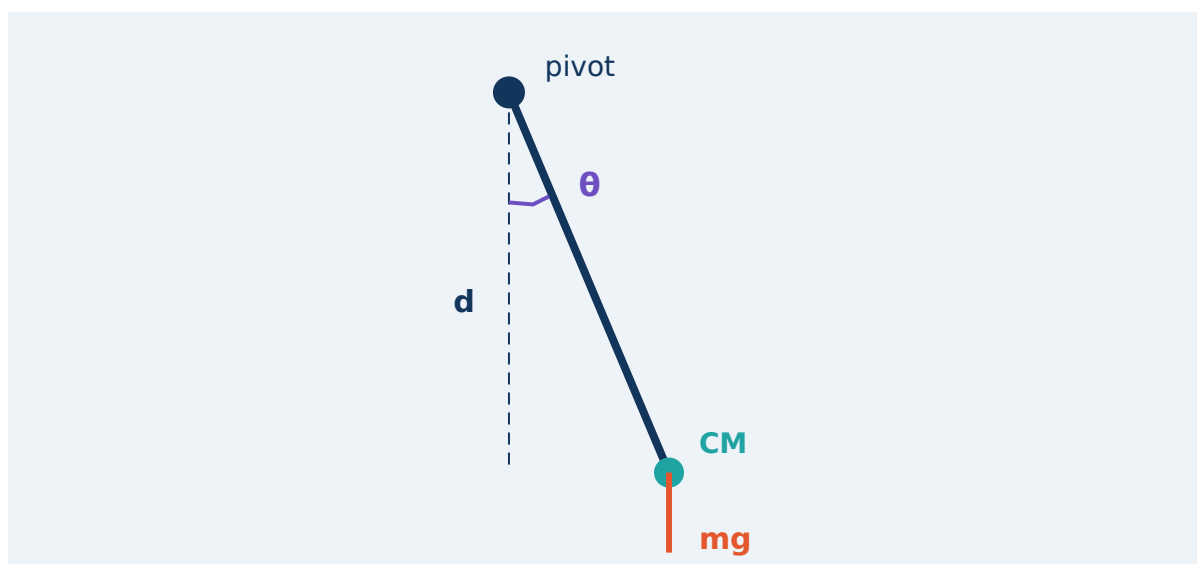
A 0.50 kg block on a spring with $k = 200$ N/m oscillates with amplitude 0.080 m. Then $\omega = \sqrt{200 / 0.50} = 20$ rad/s, so $T = 2\pi/20 = \mathbf{0.31 \text{ s}}$. The maximum speed is $A\omega = 0.080 \times 20 = \mathbf{1.6 \text{ m/s}}$, the maximum acceleration is $A\omega^2 = 0.080 \times 400 = \mathbf{32 \text{ m/s}^2}$ and the energy is $\frac{1}{2}kA^2 = \frac{1}{2}(200)(0.080)^2 = \mathbf{0.64 \text{ J}}$. Check: $\frac{1}{2}mv^2_{\text{max}} = \frac{1}{2}(0.50)(1.6)^2 = 0.64 \text{ J}$.

How does a physical pendulum work?

A physical pendulum is any rigid body that swings about a fixed horizontal pivot. If its mass is m , its centre of mass is a distance d below the pivot and its moment of inertia about the pivot is I , the torque is $\tau = -mgd \sin\theta$. For small angles, $\sin\theta \approx \theta$, and Newton's second law for rotation, $I d^2\theta/dt^2 = \tau$, becomes the SHM equation below.

Key law

$I d^2\theta/dt^2 = -mgd \theta$, so $\omega = \sqrt{(mgd / I)}$ and $T = 2\pi\sqrt{I / (mgd)}$. For a simple pendulum $I = mL^2$ and $d = L$, giving $T = 2\pi\sqrt{L/g}$. The small-angle approximation holds for amplitudes below about 10° .



Physical pendulum: the restoring torque is $-mg d \sin\theta$ about the pivot.

System	Restoring law	Angular frequency ω	Period T
Mass on a spring	$F = -kx$	$\sqrt{k/m}$	$2\pi\sqrt{m/k}$
Simple pendulum (small angle)	$\tau = -mgL \theta$	$\sqrt{g/L}$	$2\pi\sqrt{L/g}$
Physical pendulum (small angle)	$\tau = -mgd \theta$	$\sqrt{mgd/I}$	$2\pi\sqrt{I/(mgd)}$
Torsion pendulum	$\tau = -\kappa \theta$	$\sqrt{\kappa/I}$	$2\pi\sqrt{I/\kappa}$

Worked example 2

A uniform metre stick is pivoted at one end. Then $d = 0.50$ m and $I = \frac{1}{3}ML^2 = \frac{1}{3}M(1.00)^2$, so $T = 2\pi\sqrt{(\frac{1}{3}M / (M \times 9.81 \times 0.50))} = 2\pi\sqrt{0.06796} = \mathbf{1.64 \text{ s}}$. A simple pendulum 1.00 m long would take 2.01 s, so the stick swings like a simple pendulum of length $2L/3 = 0.667$ m.

Common mistake

Do not use $T = 2\pi\sqrt{L/g}$ for a rod, a disc or any extended object. It applies only to a point mass on a light string. For a physical pendulum, the correct d is the distance from the pivot to the centre of mass, and I must be taken about the pivot using the parallel-axis theorem.

Lina's tip

To prove SHM, never guess a formula. Write the restoring force or torque, put it into Newton's second law, and check that you get $d^2q/dt^2 = -(\text{constant}) q$. The square root of that constant is ω .

Key points to remember

- SHM: $d^2x/dt^2 = -\omega^2x$; $x = A \cos(\omega t + \phi)$, $v = -A\omega \sin(\omega t + \phi)$, $a = -\omega^2x$.
- Spring: $\omega = \sqrt{k/m}$; $T = 2\pi\sqrt{m/k}$; $E = \frac{1}{2}kA^2$ (constant).
- $v_{\max} = A\omega$ and $a_{\max} = A\omega^2$; v is largest at $x = 0$ and zero at $x = \pm A$.

- Physical pendulum (small angles): $\omega = \sqrt{mgd/I}$, $T = 2\pi\sqrt{I/(mgd)}$.
- Amplitude and phase come from the initial conditions; T does not depend on amplitude.

Simple harmonic motion and the physical pendulum: frequently asked questions

Why does the period of a spring not depend on amplitude?

Because the restoring force is proportional to displacement, a larger swing brings a proportionally larger force and a proportionally larger average speed. The two effects cancel, leaving $T = 2\pi\sqrt{m/k}$ for any amplitude.

What is the difference between ω and f ?

Angular frequency ω is measured in rad/s and equals $2\pi f$, where f is the frequency in hertz. The angular form is used in $x = A \cos(\omega t + \phi)$ because the cosine takes its argument in radians.

Where is the speed greatest in SHM?

At the equilibrium position, where the potential energy is lowest and the kinetic energy is at its maximum $\frac{1}{2}kA^2$. The speed there is $A\omega$. At the turning points the speed is zero and the acceleration is greatest.

Is a pendulum always simple harmonic?

Only approximately. The exact restoring torque is proportional to $\sin\theta$, not θ . For amplitudes above about 10° the period grows slightly, by roughly 7 % at 60° .

How do you find A and ϕ from initial conditions?

Use $x(0) = A \cos\phi$ and $v(0) = -A\omega \sin\phi$. Then $A = \sqrt{x_0^2 + (v_0/\omega)^2}$, and ϕ follows from $\tan\phi = -v_0/(\omega x_0)$, choosing the quadrant that matches the signs.