

Drag forces and terminal velocity with calculus

AP Physics C (Mechanics and E&M, calculus-based) · Lesson

Quick answer: Terminal velocity is the steady speed at which a resistive force balances the weight, so the net force is zero. For linear drag it is mg/b and the speed follows $v = v_t(1 - e^{-t/\tau})$; for quadratic drag it is $\sqrt{mg/c}$ and $v = v_t \tanh(gt/v_t)$.

Level: AP Physics C (Mechanics and E&M, calculus-based) · **Curriculum:** AP PC:M Unit 2

What changes when a force depends on velocity?

In earlier mechanics most forces were constant or depended on position. A resistive force is different: it depends on how fast the object moves, so the acceleration keeps changing during the motion. Newton's second law then becomes a differential equation, $m dv/dt = F(v)$, and calculus is the natural tool to solve it.

Definition

A resistive (drag) force is a force exerted by a fluid that opposes the velocity of an object moving through it. Its direction is always opposite to v , and its magnitude grows with speed.

Two models cover the AP Physics C: Mechanics course. At low speed in a viscous fluid (a ball bearing in syrup) the force is linear: $F_D = bv$. At higher speed in air (a skydiver, a cricket ball) it is closer to quadratic: $F_D = cv^2$.

How do you solve motion with linear drag?

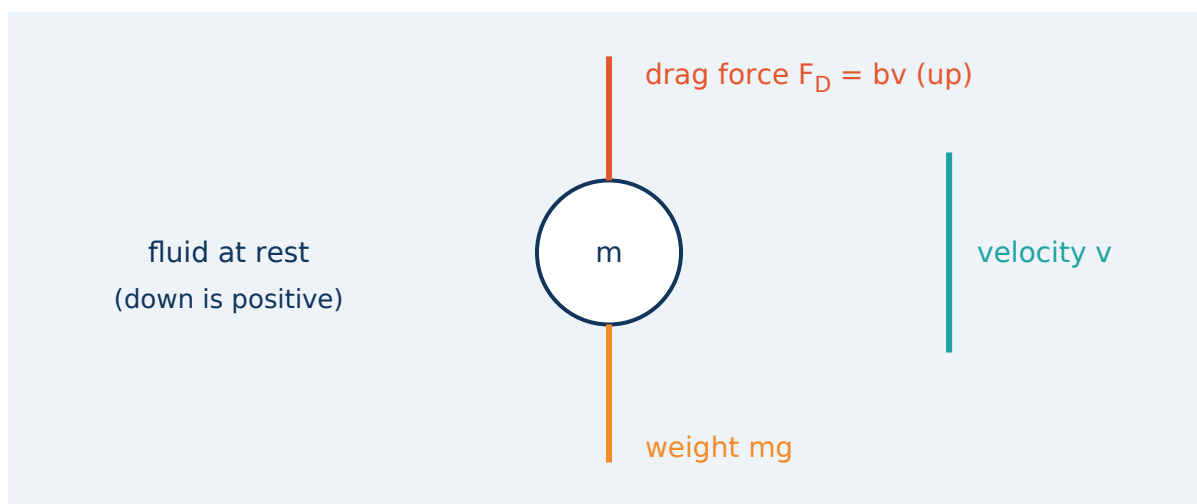
Take down as positive for a falling object released from rest. The only forces are weight and drag, so $m dv/dt = mg - bv$. Separate the variables and integrate from $v = 0$ at $t = 0$:

$$\int dv' / (mg - bv') = \int dt / m, \text{ so } -(1/b) \ln[(mg - bv)/mg] = t/m.$$

Solving for v gives an exponential approach to a limiting speed.

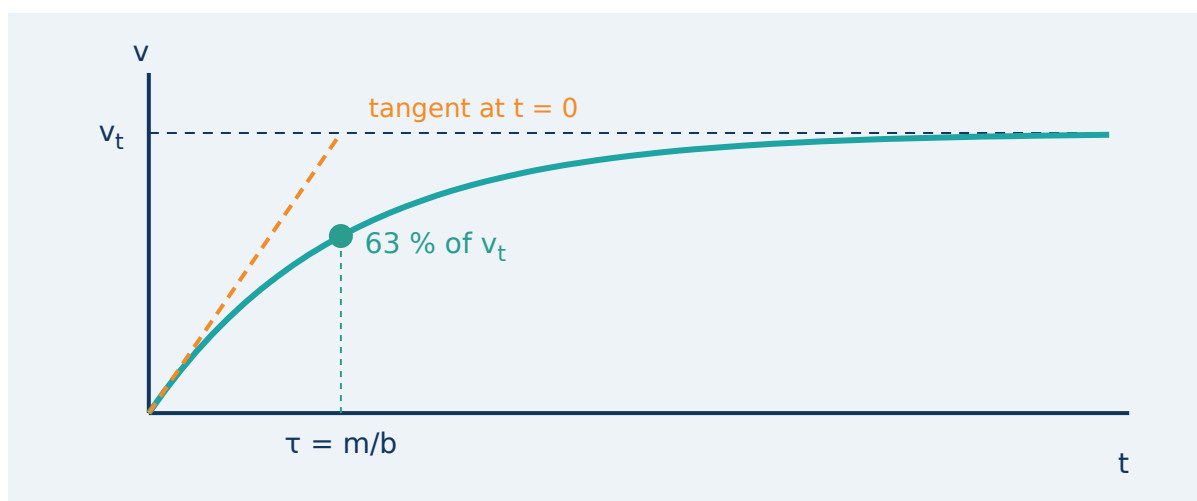
Formula

$v(t) = v_t(1 - e^{-t/\tau})$, with $v_t = mg/b$ and $\tau = m/b$. Here m is mass (kg), b the drag coefficient (kg/s), v_t the terminal speed (m/s) and τ the time constant (s).



Forces on a sphere sinking through a fluid: the drag force opposes the velocity.

Terminal speed does not need the full solution. When the speed stops changing, $dv/dt = 0$, so $mg - bv_t = 0$. Differentiating the solution gives $a = g e^{-t/\tau}$, so the acceleration starts at g and decays to zero.



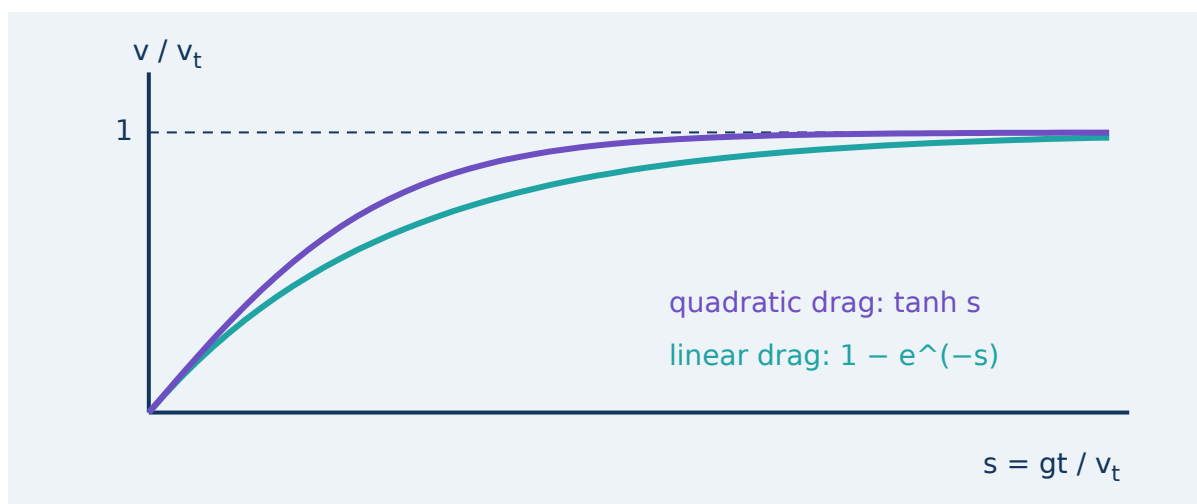
Speed against time with linear drag. The slope at $t = 0$ equals g , so the tangent reaches v_t at $t = \tau$.

Worked example 1

Ananya in Pune drops a steel ball of mass 0.050 kg into a tall jar of syrup, with $b = 0.40 \text{ kg/s}$. Find the terminal speed and the speed after 0.25 s . $v_t = mg/b = (0.050 \times 9.81)/0.40 = 1.23 \text{ m/s}$. $\tau = m/b = 0.050/0.40 = 0.125 \text{ s}$, so $t = 0.25 \text{ s}$ is 2τ . $v = 1.226 \times (1 - e^{-2}) = 1.226 \times 0.8647 = \mathbf{1.06 \text{ m/s}}$.

How does quadratic drag differ?

With $F_D = cv^2$ the equation is $m dv/dt = mg - cv^2$. Setting $dv/dt = 0$ gives $v_t = \sqrt{(mg/c)}$. Integrating $\int dv/(1 - v^2/v_t^2) = gt$ gives an inverse hyperbolic tangent, so $v = v_t \tanh(gt/v_t)$. The graph below compares both models when they share the same v_t .



Normalised speed against normalised time: quadratic drag climbs faster early on, then both approach 1.

Feature	Linear drag, $F = bv$	Quadratic drag, $F = cv^2$
Typical situation	Small, slow object in a viscous liquid	Larger, faster object in air
Terminal speed	mg/b (proportional to m)	$\sqrt{mg/c}$ (proportional to $\sqrt{m}$)
Speed against time	$v_t(1 - e^{-t/\tau})$	$v_t \tanh(gt/v_t)$
Units of coefficient	b in kg/s	c in kg/m

Worked example 2

Kofi in Accra models a skydiver of mass 80 kg with $c = 0.20$ kg/m. Find v_t and the speed after 5.0 s. $v_t = \sqrt{mg/c} = \sqrt{(784.8/0.20)} = \sqrt{3924} = 62.6$ m/s. Then $gt/v_t = (9.81 \times 5.0)/62.6 = 0.783$ and $\tanh(0.783) = 0.655$, so $v = 62.6 \times 0.655 = \mathbf{41 \text{ m/s}}$.

Common mistake

Terminal velocity does not mean there is no force. It means the net force is zero: drag has grown until it equals the weight. The speed is constant, but the object is still moving fast, and drag and weight are both large.

Lina's tip

To get a terminal speed, skip the integration: set $dv/dt = 0$ and solve the algebra. Use the full integral only when the question asks for speed at a given time.

Which signs and units should you check?

Choose one positive direction and keep it. If down is positive, drag is subtracted. If you launch an object upward, drag points down and adds to gravity. Check units: bv must be in newtons, so b is in kg/s, and cv^2 in newtons means c is in kg/m.

Key points to remember

- Linear drag: $m \, dv/dt = mg - bv$, $v_t = mg/b$, $\tau = m/b$

- Quadratic drag: $m \, dv/dt = mg - cv^2$, $v_t = \sqrt{(mg/c)}$
- At $t = \tau$ the speed is 63 % of v_t ; the acceleration is $g e^{-t/\tau}$
- Units: b in kg/s, c in kg/m
- At terminal speed $dv/dt = 0$, so drag equals weight

Drag forces and terminal velocity with calculus: frequently asked questions

What is terminal velocity?

It is the constant speed reached when the drag force equals the weight, so the net force and acceleration are zero. For linear drag it equals mg/b ; for quadratic drag it equals $\sqrt{(mg/c)}$.

Why is the approach to terminal speed exponential for linear drag?

Drag grows in proportion to speed, so the net force falls in proportion to the remaining gap $v_t - v$. A quantity whose rate of change is proportional to itself decays exponentially, with time constant $\tau = m/b$.

How do you solve $m \, dv/dt = mg - bv$?

Separate the variables to get $dv/(mg - bv) = dt/m$, integrate both sides, then rearrange. The result is $v = (mg/b)(1 - e^{-bt/m})$.

When is drag quadratic rather than linear?

Quadratic drag applies to larger objects moving fast through air, where the fluid must be pushed aside. Linear drag applies to small objects moving slowly through a viscous liquid.

What does the time constant τ mean?

It is the time for the speed to reach about 63 % of terminal speed, and after about 5τ the speed is within 1 % of it. A larger mass or smaller b gives a longer τ .