

Work as an integral and potential energy curves

AP Physics C (Mechanics and E&M, calculus-based) · Lesson

Quick answer: Work done by a variable force is the integral $W = \int F dx$, the area under the force-position graph. The net work equals the change in kinetic energy. For a conservative force, $F = -dU/dx$, so the force is the negative slope of the potential energy curve.

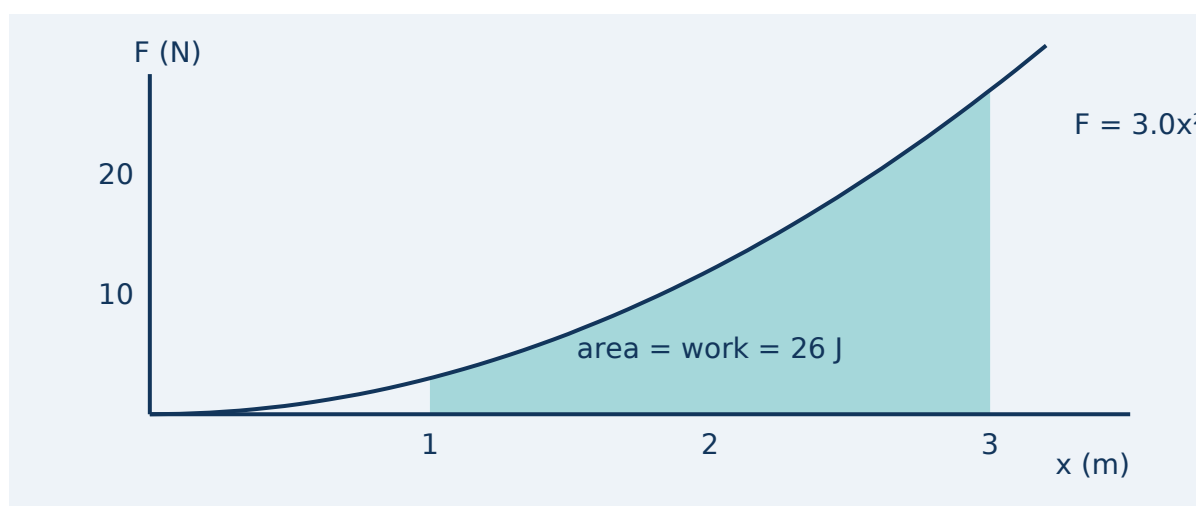
Level: AP Physics C (Mechanics and E&M, calculus-based) · **Curriculum:** AP PC:M Unit 3

How do you calculate work when the force changes?

For a constant force along the motion, $W = Fd$. When the force varies with position, split the path into tiny steps dx . On each step the force is almost constant, so the work is $F dx$, and adding them up is an integral.

Definition

Work is the energy transferred by a force acting through a displacement. For a force along the x -axis, $W = \int F(x) dx$ from x_1 to x_2 , which is the area under the F - x graph (positive where the force and displacement point the same way).



The area under a force-position graph is the work done by the force.

In general the work is the line integral of $F \cdot dr$, so only the component of force along the displacement counts. For a spring that obeys $F = -kx$, the work you must do to stretch it from rest to x is $\int kx dx = \frac{1}{2}kx^2$.

How does work change kinetic energy?

Start from Newton's second law and use the chain rule, $dv/dt = v dv/dx$. Then $F dx = m v dv$, and integrating gives the work-energy theorem.

Key law

$W_{\text{net}} = \Delta K = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$. The net work done by all forces equals the change in kinetic energy; this holds for any force, conservative or not.

Worked example 1

Aarav in Pune pushes a 2.0 kg cart with a force $F = 3.0x^2$ newtons (x in metres) from $x = 1.0$ m to 3.0 m. The cart starts at 1.0 m/s. $W = \int 3.0x^2 dx$ from 1.0 to 3.0 = $27 - 1 = 26$ J. $K_f = \frac{1}{2}(2.0)(1.0)^2 + 26 = 27$ J, so $v = \sqrt{(2 \times 27/2.0)} = \sqrt{27} = \mathbf{5.2 \text{ m/s}}$.

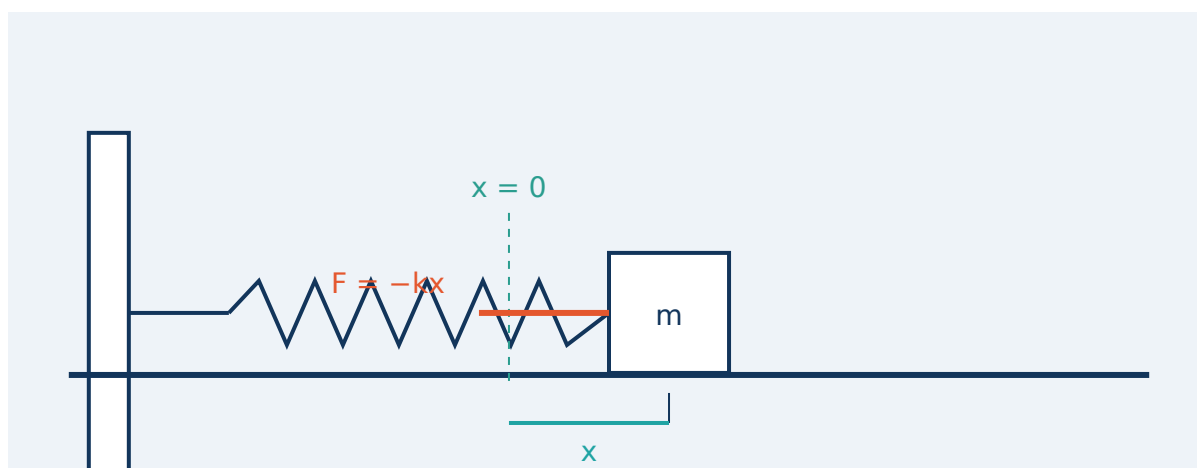
How do you get force from a potential energy function?

A force is conservative if the work it does between two points does not depend on the path. Then a potential energy $U(x)$ exists with $\Delta U = -W$, and the force is minus the slope of the U graph.

Formula

$F = -dU/dx$ and $U(x) = -\int F dx$. F is the force (N), U is potential energy (J), x is position (m). The force points toward lower U , down the slope of the graph.

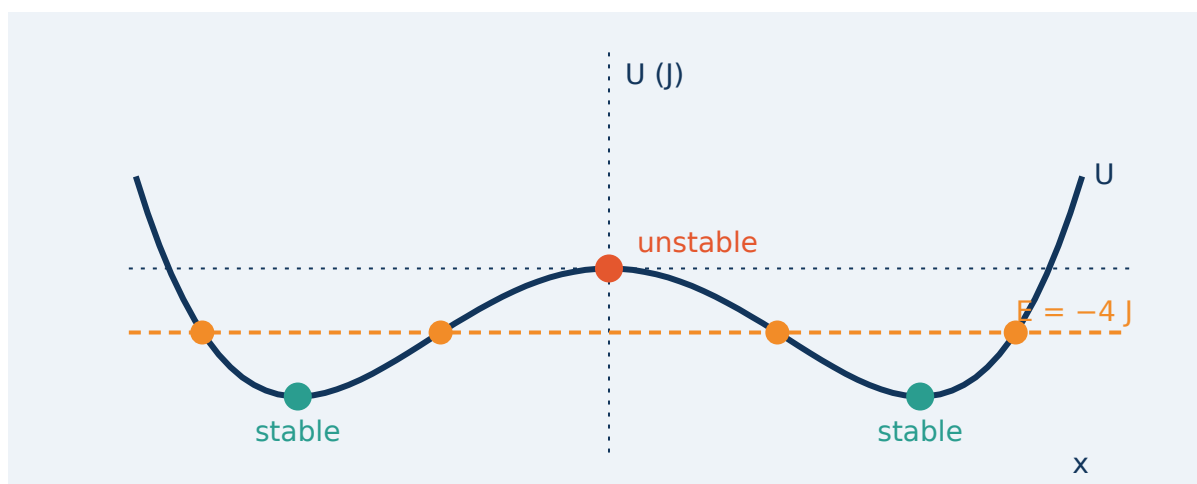
System	$U(x)$	$F = -dU/dx$
Spring	$\frac{1}{2}kx^2$	$-kx$ (towards equilibrium)
Uniform gravity	mgy	$-mg$ (downwards)
Planet's gravity	$-GMm/r$	$-GMm/r^2$ (attractive)
Quartic well	$ax^4 - bx^2$	$-4ax^3 + 2bx$



A spring stores $U = \frac{1}{2}kx^2$, and the force is the negative slope of U .

What does an energy diagram tell you?

Because the total energy is $E = K + U$ and $K \geq 0$, the particle can only be where $U \leq E$. Where $U = E$ the kinetic energy is zero, a turning point. Equilibrium occurs where $dU/dx = 0$: a minimum ($d^2U/dx^2 > 0$) is stable and a maximum is unstable.



Potential energy curve of a double well: the particle with $E = -4 \text{ J}$ is trapped in one well.

Worked example 2

Chiamaka in Lagos studies $U = 2.0x^4 - 8.0x^2$ joules for a 0.50 kg particle. $dU/dx = 8.0x^3 - 16x = 0$ gives $x = 0$ and $x = \pm\sqrt{2} = \pm 1.41 \text{ m}$. $d^2U/dx^2 = 24x^2 - 16$, which is -16 at $x = 0$ (unstable) and $+32$ at $x = \pm 1.41$ (stable). $U(\pm 1.41) = -8.0 \text{ J}$. If $E = -4.0 \text{ J}$, then at $x = 1.41 \text{ m}$, $K = -4.0 - (-8.0) = 4.0 \text{ J}$ and $v = \sqrt{(2 \times 4.0/0.50)} = \mathbf{4.0 \text{ m/s}}$.

Common mistake

Do not read the force from the height of the U graph. The force depends on the slope. Zero slope means zero force even if U is large, and the sign is the reverse of the slope: a positive slope means the force points in the negative direction.

Lina's tip

Picture a ball rolling on the shape of the U graph. It speeds up going down, slows going up, and turns around where the ball would run out of height, which is where U meets the E line.

How is power connected?

Power is the rate of doing work, $P = dW/dt = F \cdot v$. A force of 600 N acting on a motorbike at 20 m/s delivers 12 kW .

Key points to remember

- $W = \int F \, dx$; for a spring $W = \frac{1}{2}kx^2$
- Work-energy theorem: $W_{\text{net}} = \Delta K$
- Conservative force: $F = -dU/dx$
- Equilibrium: $dU/dx = 0$; stable if $d^2U/dx^2 > 0$
- Power $P = F \cdot v$; turning points occur where $U = E$

Work as an integral and potential energy curves: frequently asked questions

What is the work done by a variable force?

It is the integral of force over position, $W = \int F dx$, which equals the area under the force-position graph. The sign is positive when the force has a component along the displacement.

What is the work-energy theorem?

The net work done on an object equals its change in kinetic energy, $W_{\text{net}} = \Delta K$. It follows directly from Newton's second law, using $F dx = mv dv$.

How do you find force from potential energy?

Differentiate: $F = -dU/dx$. The minus sign means the force acts toward lower potential energy, down the slope of the graph.

How do you classify equilibrium on a U graph?

Equilibrium is at $dU/dx = 0$. If d^2U/dx^2 is positive the point is a minimum and the equilibrium is stable; if it is negative the point is a maximum and it is unstable.

What is a turning point?

It is a position where U equals the total energy E , so the kinetic energy is zero. The particle momentarily stops and then moves back into the allowed region where $U < E$.